SDEs driven by multiplicative stable-like Lévy processes

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Supercritical SDEs driven by multiplicative stable-like Lévy processes

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SDE driven by Lévy processes

This talk is concerned with strong and weak well posedness of solutions to

$$\mathrm{d}X_t = \sigma(t, X_{t-})\mathrm{d}Z_t + b(t, X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d,$$

where Z is a d-dimensional non-degenerate α-stable-like process with $\alpha \in (0, 2)$, beyond Lipschitz condition on b.

SDE driven by Brownian motion

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d,$$

Infinitesimal generator:

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b(x)\cdot\nabla.$$

1. $d = 1$ and $b = 0$: (Strong solution and PU) Yamada-Watanabe condition (1971). $\sigma(x) \in C^{1/2}(\mathbb{R})$. Counter-example: Barlow (1992) for $\sigma(x) \in C^{\beta}(\mathbb{R})$ with $\beta < 1/2$.
2. $d \geq 2$: (Strong solution and PU) Yamada-Watanabe (1971): $b = 0$. Slightly weaker than Lipschitz with a logarithmic term.
   $\sigma = I$: bounded b: Zvonkin (1974), Veretennikov (1979)
   $\sigma = I_{d\times d}$: $b(t,x)$ in L^p/L^q, Krylov and Röckner (2005).

SDE driven by Brownian motion

• Weak existence and uniqueness:

Stroock-Varadhan (1969), Krylov (1969, 1973).
$\sigma(x)$: continuous and uniformly elliptic.

Girsanov: removing and adding drifts.

Bass-C. (2003): $\sigma(x) = I_{d\times d}$, measure-valued drift $\vec{\mu}(x)\cdot\nabla$.

SDE driven by stable processes

$$dX_t = \sigma(X_{t-})dZ_t + b(X_t)dt, \quad X_0 = x \in \mathbb{R}^d,$$

where Z is an isotropic α-stable process with $\alpha \in (0, 2)$.
Infinitesimal generator when $\sigma = I_{d\times d}$:

$$\mathcal{L} = -(-\Delta)^{\alpha/2} + b \cdot \nabla.$$

Subcritical: $\alpha > 1$; critical: $\alpha = 1$; supercritical: $\alpha < 1$.

1. $d = 1$ and $\sigma = 1$ (Strong solution and PU)
   Tanaka-Tsuchiya-Watanabe (1974): $b \in L^\infty(\mathbb{R})$ when $\alpha \in (1, 2)$; $b \in C(\mathbb{R})$ when $\alpha = 1$. Counter-example for $b \in C^\beta(\mathbb{R})$ with $\beta < 1 - \alpha$ when $\alpha \in (0, 1)$.
2. $d = 1$ and $b = 0$ (Strong solution and PU)
   Komatsu (1982), Bass (2003): $\sigma \in C^{1/\alpha}(\mathbb{R})$ for $\alpha > 1$.
   Counter-example: Bass-Burdzy-C. (2004) for $\sigma \in C^\beta(\mathbb{R})$ with $\beta < 1 \wedge (1/\alpha)$ for any $\alpha \in (0, 2)$.

SDE driven by stable processes

$d \geq 2$: $\sigma = I_{d\times d}$ (Strong solution and PU)

1. Priola (2012): Z non-degenerate symmetric α-stable with $\alpha \in [1, 2)$, , $b(x) \in C^\beta(\mathbb{R}^d)$ with $\beta > 1 - (\alpha/2)$.
   Open problem for supercritical case: $\alpha \in (0, 1)$.
2. X. Zhang (2013): $\alpha \in (1, 2)$, $b(t, x)$ in some fractional Sololev space.
3. C.-Song-Zhang (2018): Z a class of Lévy processes and subordinate BMs

   - When Z is an isotropic α-stable process on $\mathbb{R}^d$, for $b \in L^\infty([0, T], C^\beta(\mathbb{R}^d))$ with $\beta > 1 - (\alpha/2)$ for $\alpha \in (0, 2)$.
   - When Z is a cylindrical α-stable process on $\mathbb{R}^d$, for $b \in L^\infty([0, T], C^\beta(\mathbb{R}^d))$ with $\beta > 1 - (\alpha/2)$ for $\alpha > 2/3$.

SDE driven by stable processes

Weak existence and weak uniqueness.

1. Z: isotropic α-stable processes with $\alpha \in (1, 2)$, $\sigma = I_{d\times d}$:

   • Portenko (1994 for $d = 1$), Podolynny-Portenko (1995 for $d \geq 2$): $b(x) \in L^p(\mathbb{R}^d)$ with $p > d/(\alpha - 1)$.

   • C.-L. Wang (2016): $b(x)$ in Kato class, including $L^\infty(\mathbb{R}^d) + L^p(\mathbb{R}^d)$ with $p > d/(\alpha - 1)$.

2. Z: isotropic α-stable processes with $\alpha \in (0, 1)$, $\sigma = I_{d\times d}$:
   • Tanaka-Tsuchiya-Watanabe (1974), Tsutsumi (1974): $d = 1$. Counter-example for $b \in C^\beta(\mathbb{R})$ with $\beta < 1 - \alpha$;

   • Kulik (2019): $0 < c_1 \leq \sigma(x) \leq c_2$ scalar Hölder, $b(x) \in C^\beta(\mathbb{R}^d)$ with $\beta > (1 - \alpha)^+$.

3. Zhao (2019): a subclass of α-stable processes Z with $\alpha \in (0, 1)$, $\sigma = I_{d\times d}$, $b(x) \in C^\beta(\mathbb{R}^d)$ with $\beta > 1 - \alpha$.

Our setting

Z: Lévy process with $\mathbb{E}\left[e^{i\xi\cdot(Z_t-Z_0)}\right]=e^{-t\psi(\xi)}$:

$$\psi(\xi)=\int_{\mathbb{R}^d}\left(1-e^{i\xi\cdot z}+i\xi\cdot z\mathbb{1}_{\{|z|<1\}}\right)\nu(dz).$$

Lévy measure ν: $\int_{\mathbb{R}^d}(1\wedge|z|^2)\nu(dz)<\infty$.

For $\alpha\in(0,2)$, let $\mathbb{L}_{non}^{(\alpha)}$ be the space of all non-degenerate α-stable Lévy measures $\nu^{(\alpha)}$:

$$\nu^{(\alpha)}(A)=\int_0^\infty\left(\int_{\mathbb{S}^{d-1}}\mathbb{1}_A(r\theta)\Sigma(\mathrm{d}\theta)\right)\frac{\mathrm{d}r}{r^{1+\alpha}},\quad A\in\mathcal{B}(\mathbb{R}^d),$$

where Σ is a finite measure on $\mathbb{S}^{d-1}$ with

$$\int_{\mathbb{S}^{d-1}}|\theta_0\cdot\theta|\,\Sigma(\mathrm{d}\theta)>0\quad\text{for every }\theta_0\in\mathbb{S}^{d-1}.$$

Stable-like Lévy process

Let Z be a purely discontinuous Lévy process with Lévy measure ν so that

$$\nu_1(A) \leq \nu(A) \leq \nu_2(A), \quad A \in \mathcal{B}(B(0,1)).$$

for some $\nu_1, \nu_2 \in \mathbb{L}_{non}^{(\alpha)}$.

Example: Z is the independent sum of α-stable and β-stable processes with $\beta < \alpha$.

Assume

$$\Lambda^{-1}|\xi| \leq |\sigma(t,x)\xi| \leq \Lambda|\xi| \quad \text{and} \quad |b(t,x)| \leq \Lambda.$$

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Our main results

Theorem (C.-Zhang-Zhao, 2021+)

Under the above assumptions, for each $x \in \mathbb{R}^d$, the SDE

$$dX_t = \sigma(t, X_{t-})dZ_t + b(t, X_t)dt, \quad X_0 = x_0.$$

1. *has a unique weak solution if $\sigma(t, x)$ and $b(t, x)$ are C^β in x with $\beta > (1 - \alpha)^+$;*
2. *has a unique strong solution if $\sigma(t, x)$ is Lipschitz in x and $b(t, x)$ are C^β in x with $\beta > 1 - (\alpha/2)$.*

- Hold for any α-stable process including cylindrical ones.
- Multiplicative noise; Localization
- Sharp in σ (strong solution); sharp in b (weak solution).

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Our approach

The infinitesimal generator for SDE is $\mathcal{L}_t + b(t,x)\cdot\nabla$, where

$$\mathcal{L}_t u(x) := \int_{\mathbb{R}^d} \big(u(x+\sigma(t,x)z) - u(x) - \mathbb{1}_{\{|z|\le 1\}}\sigma(t,x)z\cdot\nabla u(x)\big)\,\nu(\mathrm{d}z).$$

For strong well posedness, we use Zvonkin's change of variable to remove the drift. Key: existence, uniqueness and regularity estimates for

$$\partial_t u = (\mathcal{L}_t - \lambda)u + b\cdot\nabla u + f \quad \text{with } u(0,x)=0.$$

We show when $\alpha+\beta>1$ and $\sigma(t,x)=\sigma(t)$, for every $p>d/(\alpha+\beta-1)$,

$$\|u\|_{L^\infty([0,T];B^{\alpha+\beta}_{p,\infty})} \le C\|f\|_{L^\infty([0,T];B^{\beta}_{p,\infty})},$$

and for any $\gamma\in(0,\alpha+\beta)$,

$$\|u\|_{L^\infty([0,T];B^{\gamma}_{p,\infty})} \le c_\lambda\|f\|_{L^\infty([0,T];B^{\beta}_{p,\infty})},$$

where $B^\beta_{p,\infty}$ is the usual Besov space and $c_\lambda\to 0$ as $\lambda\to\infty$.

Our approach for strong well posedness

For general Hölder $\sigma(t, x)$, we use a localization and a patching-together procedure to establish the above a priori estimate for Lévy measure ν with bounded support.

Take $f = b$ and $\lambda > 0$ large. By Sobolev embedding, $\|\nabla u\|_\infty < 1/2$. So $\Phi(t, x) := x + u(t, x)$ is 1-1. When $\sigma(t, x)$ is Lipschitz in x, $\beta > 1 - (\alpha/2)$ and ν has bounded support, by Ito's formula, $Y_t := \Phi(t, X_t)$ satisfies an SDE with Lipschitz coefficients. Thus Y_t is well-posed and so is X_t.

General ν: truncation and piecing-together argument.

New feature: we use the Littlewood-Paley theory and some Bernstein's type inequalities to establish the above a priori estimates for the fractional PDE. This approach allows us to address the open problem affirmatively for any non-degenerate stable processes.

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Littlewood-Paley theory

(i) a substitute for orthogonality arguments in L^2 spaces.

(ii) decompose f into a sum of functions $\Pi_j f$ with localized frequencies (between 2^{j-1} and $3 \cdot 2^{j-1}$), and use it to characterize the Besov spaces.

(iii) Bernstein's type inequality: estimates on $\|\nabla^k \Pi_j f\|_q$ and $\|(-\Delta)^{\beta/2} \Pi_j f\|_p$.

(iv) Commutator estimates on $\|[\Pi_j, f\,]\, g\|_p$.

(iv) Sobolev embedding theorem for Besov spaces.

Weak well posedness

(i) Uniqueness for the martingale problem for SDE driven by truncated Lévy process, using solution of fractional PDE.

(ii) Weak existence follows from a weak convergence argument.

(iii) Weak uniqueness for general ν: resurrect at times $\{\tau_k; k \geq 1\}$ when the driving Lévy process Z makes jumps larger than 1 and show the resulting solution satisfies SDE driven by the truncated Lévy process. This gives weak uniqueness on $[0, \tau_1)$, and then on $[0, \tau_2)$, ...

Thanks for watching!